Tether equilibria in a linear parallel force field

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1 Introduction

A tether, being in equilibrium in a parallel force filed, lies in a plane parallel to the lines of force [1]. In this plane we imposed an orthogonal coordinate system Oxz with the "vertical" axis Oz parallel to the force line. The projection of the tension of the tether upon a line perpendicular to the vertical axis is constant. In a uniform force filed, the shape of a hanging tether is known to be given, up to translation and reflection, by

$$z = c_1 \cosh\left(c_1^{-1} x\right), \quad c_1 > 0.$$

In our case of a linear parallel force we regard the magnitude of the force proportional to the distance to the "horizontal" Ox axis. We consider both the attracting as well as the repelling case. The equilibria of the hanging tether in a linear parallel force field are given, up to translation along the horizontal axis and reflection, by

$$z = \begin{cases} z_* = \sqrt{\dfrac{c_2}{2}\left(\mathcal{R}_\beta\left(\dfrac{x}{\sqrt{c_2}}\right) + \mathcal{R}_\beta{}^{-1}\left(\dfrac{x}{\sqrt{c_2}}\right)\right) + \lambda}, & \pm\lambda \neq c_2 > 0, \\ \\ \dfrac{\sqrt{\lambda^2 - c_2^2}}{z_*}, & \lambda > c_2 > 0, \end{cases}$$

separated into classes and subclasses by periodical

$$z = \sqrt{2\,c_2}\,\sec\left(\sqrt{\frac{2}{c_2}}\,x\right), \quad c_2 > 0,$$

vertical (parallel to the Oz axis) and horizontal axial (along the Ox axis) solutions. Here R_β is a second order elliptic function with a pole at zero, whose values are 0, $\beta = c_2^{-1}\lambda + \sqrt{c_2^{-2}\lambda^2 - 1}$ and $\beta^{-1} = c_2^{-1}\lambda - \sqrt{c_2^{-2}\lambda^2 - 1}$ at the points, where the derivative vanishes. The period parallelogram of the function R_β is a rectangle when $\lambda^2 > c_2^2 > 0$, and is a rhombus when $\lambda^2 < c_2^2 > 0$. The classification of the equilibria forms is based upon the relationship of the parameters λ and c_2.

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2 Isoperimetrical statement of the problem and tether equilibria as conditional extremals

Consider an absolutely flexible, uniform, non-stretchable tether whose length is l and whose ends are fixed in a linear parallel force field. In the plane of the tether we introduce an orthogonal coordinate system Oxz. The Oz axis, parallel to the force lines, we call *the vertical axis*, and the perpendicular to it the Ox axis we call *the horizontal axis*. The coordinates of the ends of the tether we denote as (x_0, z_0) and (x_1, z_1). Unless stated otherwise we presume that $x_0 < x_1$, and that the shape of the tether is being given as a graph of single-valued analytic function $z = z(x)$. We introduce, correspondingly, the length functional and the potential of the external forces, acting on the tether,

$$L := L(z, z') := \int_{x_0}^{x_1} \sqrt{1 + z'^2}\, dx, \quad U := U(z, z') := \int_{x_0}^{x_1} z^2 \sqrt{1 + z'^2}\, dx,$$

where $(\cdot)'$ denotes differentiation with respect to x.

The equilibria forms of the tether are defined as the extremals of the potential among the curves $z(\cdot)$ satisfying the boundary conditions

$$z(x_0) = z_0, \ \ z(x_1) = z_1 \tag{1}$$

and the constraint condition

$$L = l. \tag{2}$$

By *solution* we shall not merely imply a solution on the closed interval $[x_0,\ x_1]$, but its continuation as long as it exists.

3 The differential equations of the extremals and their mechanical interpretation

The external force acting on the tether has no horizontal component, hence the horizontal component of the tension is constant, which we shall denote by τ. Solutions with nonzero horizontal tension projection we shall call *general position solutions*. They are extremals of the integral

$$U - \lambda L = \int_{x_0}^{x_1} F(z, z')\, dx, \ \ F(z, z') := (z^2 - \lambda)\sqrt{1 + z'^2},$$

and they satisfy the Euler-Lagrange equation

$$\frac{\partial}{\partial z} F(z, z') - \frac{d}{dx}\left(\frac{\partial}{\partial z'} F(z, z')\right) = 0, \tag{3}$$

where the undetermined Lagrange multiplier λ is calculated using the constraint condition (2) for the given boundary conditions (1).

Since the integrand $F(z, z')$ does not explicitly depend upon x, the equation (3) can be integrated to yield

$$F(z, z') - z' \frac{\partial}{\partial z'} F(z, z') = c_2, \tag{4}$$

where c_2 is constant.

In our case equation (4), after simplification, becomes

$$z^2 - \lambda = c_2 \sqrt{1 + z'^2}. \tag{5}$$

Note that the differential equation (5) is invariant with respect to the sign conversion $z \mapsto -z$.

Solutions z, whose continuations are bounded by the inequality

$$z^2 \leq \lambda - |c_2|, \tag{6}$$

we shall call *bounded*, whereas solutions, which satisfy the inequality

$$z^2 \geq \lambda + |c_2|, \tag{7}$$

we shall call *unbounded*.

3.1 The repelling and the attracting cases

In order to display the mechanical interpretation of equations(5), let σ denote the external force density per unit length acting on an element located a unit away from the horizontal axis. The equilibrium equation for an infinitely small element is

$$\tau \, dz' + \sigma \sqrt{1 + z'^2} \, z \, dx = 0.$$

And assuming the non-vanishing of the horizontal component of the tension, it can be rewritten as a second order differential equation

$$z'' = -\tau^{-1} \sigma \sqrt{1 + z'^2} \, z,$$

which can be arrived at by differentiating the first order equation(5) with

$$c_2 = -2\sigma^{-1}\tau.$$

The case of $\sigma^{-1}\tau > 0$, corresponding to the case of $c_2 < 0$, we shall call *the repelling case*, and the case of $c_2 > 0$ we call *the attracting case*.

We note that in either case

$$z'' = 0 \quad \Leftrightarrow \quad z = 0;$$

and that the signs of z'' and z are opposite to each other in the repelling case, whereas they coincide in the attracting case.

4 Regular and special solutions

Equation (5) leads to equation

$$z'^2 = c_2^{-2}(z^2 - \lambda)^2 - 1. \tag{8}$$

A differential equation of type (8) for $0 < |c_2| < \infty$ we shall call an *equilibrium equation*; and solutions of equilibrium equations, aside from horizontal non-axial solutions, we shall call *regular solutions* if $\lambda^2 \neq c_2^2$, and *special solutions* if $\lambda^2 = c_2^2$.

Fix $c_2 > 0$. Let h be the cubic polynomial

$$h(t) := t^3 - h_2 t - h_3, \;\; h_2 := \frac{3\alpha^2 - 1}{c^2}, \;\; h_3 := \frac{\alpha(1 - 2\alpha^2)}{c^3}, \;\; \alpha := \frac{2\lambda}{3c_2}, \;\; c := 2c_2. \tag{9}$$

The substitution

$$z^2 = \frac{c_2\left(q + q^{-1}\right)}{2} + \lambda, \tag{10}$$

with the subsequent linear substitution

$$q = cp - \alpha, \tag{11}$$

transforms the differential equation (8) into the "classical" Weierstrass differential equation

$$p'^2 = 4\left(p^3 - \frac{3\alpha^2 - 1}{c^2}p - \frac{\alpha(1 - 2\alpha^2)}{c^3}\right) = 4\left(p^3 - h_2 p - h_3\right) = 4h(p). \tag{12}$$

Let $g_2 := 4h_2$, $g_3 := 4h_3$. If the discriminant D of the cubic h

$$D := g_2^3 - 27g_3^2 = \frac{4d^2}{c^6}, \;\; d := d(\alpha) := \sqrt{\left(\frac{3\alpha}{2}\right)^2 - 1} = \sqrt{\left(\frac{\lambda}{c_2}\right)^2 - 1}, \tag{13}$$

distinct from zero, then a solution to equation (12) is Weierstrass elliptic function $\wp$

$$p(x) = \wp(x) := \wp(x; \Lambda) := x^{-2} + \sum_{y \in \Lambda'} (x - y)^{-2} - y^{-2}$$

with doubly-periodic lattice

$$\Lambda := \Lambda(\alpha) := \Lambda(\alpha, c) := \mathbb{Z}w_+ \oplus \mathbb{Z}w_-, \;\; w_+ w_-^{-1} \in \mathbb{C} \setminus \mathbb{R},$$

satisfying the relations

$$h_2 = 15 \sum_{y \in \Lambda'} y^{-4}, \;\; h_3 = 35 \sum_{y \in \Lambda'} y^{-6}, \;\; \Lambda' := \Lambda \setminus \{0\}.$$

Weierstrass function $\wp$ has a double pole at zero and a Laurant's series expansion about zero

$$\wp(x) = x^{-2} + \sum_{k=2}^{\infty} a_k x^{2k-2}, \;\; a_2 = \frac{h_2}{5}, \;\; a_3 = \frac{h_3}{7}, \;\; a_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} a_m a_{k-m} \;\; \forall k > 3.$$

Translating the domain of definition of Weierstrass function we obtain a general nontrivial solution of Weierstrass differential equation (12)

$$p(x) = \wp(x - \bar{x}) \tag{14}$$

with a pole in $\bar{x}$.

For definitive choice of the parameter d and subsequent parameters we agree on picking the square root in formula (13) from the right half-plane without the negative imaginary axis. The cubic $h(p) = p^3 - h_2 p - h_3$ decomposes into linear factors

$$h(p) = \left(p - \frac{\alpha - \beta}{c}\right)\left(p - \frac{\beta - 2\alpha}{c}\right)\left(p - \frac{\alpha}{c}\right), \ \beta := \frac{3\alpha}{2} + d.$$

The set of solutions of equation (12) for a fixed value of the parameter α is invariant under the action of the subgroup, of linear fractional transformations, isomorphic with the four element group $\mathbb{Z}_2 \times \mathbb{Z}_2$, any two, of the three nontrivial elements of which, can serve as generators. We shall denote this group by $\mathcal{F}$ and we indicate its three nontrivial elements

$$T = \begin{pmatrix} \alpha & \dfrac{1-\alpha^2}{c} \\ c & -\alpha \end{pmatrix}, \ T_+ = \begin{pmatrix} \alpha - \beta & \dfrac{\alpha(2\beta - \alpha) - 1}{c} \\ c & \beta - \alpha \end{pmatrix}, \ T_- = \begin{pmatrix} \beta - 2\alpha & \dfrac{\alpha(5\alpha - 2\beta) - 1}{c} \\ c & 2\alpha - \beta \end{pmatrix}.$$

We shall identify a fractional linear transformation with its matrix as an element of the projective special linear group $PSL(2, \mathbb{C})$. The trace of each of the three indicated matrices is zero.

In other wards, the Weierstrass differential equation (12) is invariant under the action of the group $\mathcal{F}$ of linear fractional transformations, the elements of which are I, T, T_+ T_-, where I is the identity transformation. Choosing T T_+ as generators, we write down the generating relations for $\mathcal{F}$

$$T \circ T = I, \ T \circ T_+ = T_-.$$

We introduce the inversion operator Q and two, dependent upon the parameter γ, operators – the multiplication operator $M(\gamma)$ and the shift operator S^γ

$$Q := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ M(\gamma) := \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \ S^\gamma := \begin{pmatrix} 1 & c^{-1}\gamma \\ 0 & 1 \end{pmatrix},$$

and we point out their elemetary composition properties

$$M(\gamma) \circ Q \circ M(\gamma) = Q, \ M(\gamma_1) \circ M(\gamma_2) = M(\gamma_1\gamma_2), \ S^{\gamma_1} \circ S^{\gamma_2} = S^{\gamma_1+\gamma_2}. \tag{15}$$

We define an conjugation action of an invertible operator N upon an arbitrary operator A

$$N \cdot A := N \circ A \circ N^{-1}.$$

A group, conjugate to the group $\mathcal{F}$ under the action of N we denote $N \cdot \mathcal{F}$. The group $N \cdot \mathcal{F}$ is isomorphic with the group $\mathcal{F}$. Denote by R, R_+ R_- – the elements of $S^{-\alpha} \cdot \mathcal{F}$, corresponding to the elements T, T_+ T_- of the group $\mathcal{F}$,

$$R = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}, \quad R_+ = \begin{pmatrix} -\beta & -c^{-1} \\ c & \beta \end{pmatrix}, \quad R_- = \begin{pmatrix} -\beta^{-1} & -c^{-1} \\ c & \beta^{-1} \end{pmatrix}.$$

Let p be any, not necessarily coinciding with the Weierstrass function $\wp$, solution of equation (12). We introduce the operator

$$K : p \mapsto I(p) + R(p) = p + c^{-2} p^{-1}.$$

The operator K is not fractional linear; it satisfies the relations

$$K \circ R = K, \ \ K \circ R_+ = K \circ R_- = S^{-3\alpha} \cdot \big(M(c_2^{-2} d^2) \circ Q \big) \circ K. \tag{16}$$

The operator

$$Z := M(c_2^2) \circ S^{3\alpha} \circ K \circ S^{-\alpha}$$

maps a solution p of equation (12) to a function z^2 – the square of a solution of equation (8). Furthermore, since properties (16) and (15) hold, we have

$$Z \circ T = Z, \ \ Z \circ T_+ = Z \circ T_- = M(c_2^2 d^2) \circ Q \circ Z.$$

Thereby, the differential equation (8) turns out being invariant under the transformation

$$\Theta := M(c_2 d) \circ Q : z \mapsto z_* = \sqrt{\lambda^2 - c_2^2}\, z^{-1},$$

and since

$$\frac{z_*''}{z_*} = \frac{2z'^2 - z'' z}{z^2},$$

then $z'' z > 0$ implies that $z_*'' z_* < 0$, in a neighborhood of an extremum, and a solution z in the repelling case transforms into a solution z_* in the attracting case. A bounded,in the sense of inequality (6), solution z transforms into an unbounded,in the sense of inequality (7), solution z_*. We call the operator Θ – *the transformation operator*; and we note that it exists for $\lambda^2 \geq c_2^2$, and is injective for $\lambda^2 > c_2^2$.

Let $z(\alpha, c)$ denote a solution of the equilibrium equations for given parameters α and c. On the set of solutions of the equilibrium equations we introduce a *homothety operator* $\Phi = \Phi(\gamma)$, dependent upon a parameter γ, via describing its action on an element $z = z(\alpha, c)$ of the solution set

$$\left(\Phi(\gamma) \cdot z\right)(x) = \gamma\, z(\gamma^{-1} x).$$

Since

$$\Phi(\gamma_2) \cdot (\Phi(\gamma_1) \cdot z) = \Phi(\gamma_2 \gamma_1) \cdot z,$$

the set of homothety operators can be endowed with a group structure, isomorphic with the multiplicative group of the invertible elements of the field $\mathbb{C}$.

We point out that

$$\Phi(i) \cdot z(\alpha, c) = z(-\alpha, c),$$

and the equilibrium equations are invariant with respect to the action of the operator $\Phi(-1)$, although the action of the latter is nontrivial, namely

$$(\Phi(-1) \cdot z)(x) = -z(-x).$$

And if $\gamma > 0$, then

$$\Phi(\sqrt{\gamma}) \cdot z(\alpha, c) = z(\alpha, \gamma c).$$

4.1 Special solutions

For $\lambda = c_2$ the equilibrium equation (8) takes the form

$$z'^2 = \frac{z^2}{c_2}\left(\frac{z^2}{c_2} - 2\right),$$

whose solutions, aside from the horizontal non-axial solutions $z \equiv \pm\sqrt{c}$, are bounded in the sense of inequality (6) horizontal axial solution

$$z \equiv 0,$$

which we shall call *real horizontal axial solution*, and unbounded special solutions

$$z(x) = \pm\sqrt{c}\ \sec\left(2\sqrt{c^{-1}}(x - x_2)\right), \tag{17}$$

where x_2 is an extremum point.

For $\lambda = -c_2$ the equilibrium equation (8) takes the form

$$z'^2 = \frac{z^2}{c_2}\left(\frac{z^2}{c_2} + 2\right),$$

whose solutions are imaginary

$$z \equiv 0, \;\; z \equiv \pm\, i\sqrt{c}, \;\; z(x) = \pm\, i\sqrt{c}\,\mathrm{sech}\left(2\sqrt{c^{-1}}(x - x_2)\right).$$

The horizontal axial solution, for $\lambda = -c_2$, which we shall call *imaginary horizontal axial solution*, is not bounded in the sense of inequality (6), and is unbounded in the sense of inequality (7).

5 Marginal solutions

By *marginal solutions* we shall imply solutions, composed of two open vertical rays, emanating from x_0 and x_1, and the close interval $[x_0,\, x_1]$ connecting them on the horizontal axis, on which the tension vanishes. Unless otherwise indicated, we assume that

$$-\infty < x_0 < x_1 < +\infty.$$

We distinguish two kinds of marginal solutions: *one-sided marginal solutions*, corresponding to vertical rays whose directions coincide, and *two-sided marginal solutions*, corresponding to vertical rays whose directions are opposite to each other.

Vertical solutions can be viewed as special cases of marginal solutions when

$$-\infty < x_0 = x_1 < +\infty,$$

separating the one-sided marginal solutions from the two-sided.

The marginal horizontal axial solution shall be the separating common limit solution of the one-sided and the two-side marginal solutions when

$$-\infty = x_0 < x_1 = +\infty.$$

This solution is a separating common limit solution for the real and the imaginary horizontal axial solutions.

6 Limit solutions

By *limit solutions* we shall denote linear solutions of the form

$$z = dx + b, \ \ d^2 + b^2 > 0. \tag{18}$$

These solutions correspond to the limiting value $|c_2| = \infty$. Among these we distinguish solutions of the form $z \equiv z_0 \neq 0$ and call them *horizontal non-axial solutions*. Non limit solutions, we shall call *equilibrium forms*.

7 Critical values of regular solutions

Introduce the notation $\beta^+ := \beta$, $\beta^- := \beta^{-1} = 3\alpha - \beta$. The set of the fixed points of an operator A we denote by F^A and call it *the fixed set of the operator* A. The fixed set of each of the operators R, R_+ and R_- is a two-element set

$$F^R = \left\{\frac{1}{c}, -\frac{1}{c}\right\}, \ \ F^{R_\pm} = \left\{\frac{\sqrt{\beta^{\pm 2} - 1} - \beta^\pm}{c}, \frac{-\sqrt{\beta^{\pm 2} - 1} - \beta^\pm}{c}\right\}.$$

Define the action of an operator A upon a set F by

$$A(F) := \{A(f) : f \in F\}.$$

Since

$$F^{S^\gamma \cdot A} = S^\gamma(F^A) = \left\{f + \frac{\gamma}{c} : f \in F^A\right\} \tag{19}$$

then

$$F^T = F^{S^\alpha \cdot R} = S^\alpha(F^R) = \left\{\frac{\alpha + 1}{c}, \frac{\alpha - 1}{c}\right\}$$

and

$$F^{T_\pm} = F^{S^\alpha \cdot R_\pm} = S^\alpha(F^{R_\pm}) = \left\{ \frac{\alpha - \beta^\pm + \sqrt{\beta^{\pm 2} - 1}}{c}, \frac{\alpha - \beta^\pm - \sqrt{\beta^{\pm 2} - 1}}{c} \right\}.$$

Each of the operators R, R_+ and R_- acts as a trivial permutation of the pair of its own fixed set, and acts as a nontrivial transposition of the pair of the fixed set of any of the two other operators. And since operators T, T_+ and T_- are operators conjugate to, correspondingly, the operators R, R_+ and R_- under the action of S^α then, due to (19), each of them acts as a nontrivial transposition of the pair of the fixed set of any of the two other operators.

The set F^T under the action of the operator Z is mapped to a two-element set

$$Z(F^T) = M(c_2^2) \circ S^{3\alpha} \circ K(F^R) = \left\{ \frac{c_2(3\alpha + 2)}{2}, \frac{c_2(3\alpha - 2)}{2} \right\} = \{\lambda + c_2, \lambda - c_2\},$$

and each of the two-element sets F^{T_+} and F^{T_-} is mapped to a single element set

$$Z(F^{T_\pm}) = M(c_2^2) \circ S^{3\alpha} \circ K(F^{R_\pm}) = \left\{ \frac{c_2(3\alpha - 2\beta^\pm)}{2} \right\} = \{\mp c_2 d\}.$$

Let p be a solution of equation (12), z is the corresponding solution of equation (8), x – an inner extremum point of the solution z then x is an inner extremum point of $z^2 = Z(p)$ and is an inner extremum point of $K \circ S^{-\alpha}(p)$, so then

$$p'(x)\left(1 - \frac{1}{(cp(x) - \alpha)^2}\right) = 0$$

holds, which assuming that $p'(x) \neq 0$, is equivalent to

$$(cp(x) - \alpha)^2 = 1,$$

which exactly means that

$$p(x) \in F^T.$$

7.1 Period rectangle and period rhombus

Along with $c > 0$, we fix $\alpha \geq 0$. Let $\wp$ be the corresponding Weierstrass function with the lattice $\Lambda = \Lambda(\alpha) = \mathbb{Z}w_+ \oplus \mathbb{Z}w_-$. We introduce the notation

$$\delta := \sqrt{\beta^2 - 1}, \;\; \delta_- := \sqrt{1 - \beta^{-2}}, \;\; \rho := \beta + \delta, \;\; \rho_- := \beta^- - i\delta_-,$$

and point out the relations

$$\beta^{-1}\delta = \delta_-, \;\; \delta\delta_- = 2d, \;\; \rho^{-1} = \beta - \delta, \;\; \rho_-^{-1} = \beta^- + i\delta_-.$$

We consider two cases: $\alpha > \frac{2}{3}$ and $0 \leq \alpha < \frac{2}{3}$.

If $\alpha > \frac{2}{3}$ then $d > 0$ and the cubic polynomial h, in (9), has three real roots. Moreover, $\beta > 1$, $\rho_-^{-1} = \overline{\rho}_-$, and we may assume that $w_+ > 0$ and $iw_- > 0$ since, in this case, the period parallelogram is a rectangle whose sides parallel the real and imaginary axes. If $0 \leq \alpha < \frac{2}{3}$ then $-1 \leq d^2 < 0$ and the cubic polynomial h has one real and two complex conjugate roots. Here $\beta^{-1} = \overline{\beta}$, $\overline{\rho}_- = \rho$, and the period parallelogram, in this case, is a rhombus whose diagonals parallel the real and imaginary axes. In choosing a basis we shall assume that $w_+ + w_- > 0$ and $i(w_+ - w_-) > 0$.

Whatever the case, the elliptic function

$$\mathcal{R}(x) := c\wp(\sqrt{c}x) - \alpha$$

satisfies the differential equation

$$\mathcal{R}'^2 = 4\mathcal{R}\left(\mathcal{R} + \beta\right)\left(\mathcal{R} + \beta^{-1}\right).$$

Let $\Lambda_2 := \frac{1}{2}\Lambda$ and let $K := \mathcal{R}(\mathbb{C}\,/\sqrt{c^{-1}}\Lambda_2)$ be the image under the mapping $\mathcal{R}$ of the half-period parallelogram $\mathbb{C}\,/\sqrt{c^{-1}}\Lambda_2$. The period parallelogram of the function $\mathcal{R}$ can be divided into four half-period parallelograms, on which the function $\mathcal{R}$ maps to K, $R(K)$, $R_+(K)$ $R_-(K)$. We call "*central*" the case where the period parallelogram turns out being a square, and we shall represent such a division in two central cases in fig. 1. The case $\alpha = \sqrt{2^{-1}}$ corresponds to a square with two sides parallel to the real axis, and the case $\alpha = 0$ corresponds to a square with one of its diagonals parallel to the real axis.

Let

$$\Lambda_4 := \frac{1}{2}\Lambda_2 = \frac{1}{4}\Lambda, \ \ \Delta := 2(d + \delta), \ \ \Delta_- := 2(d + i\delta_-).$$

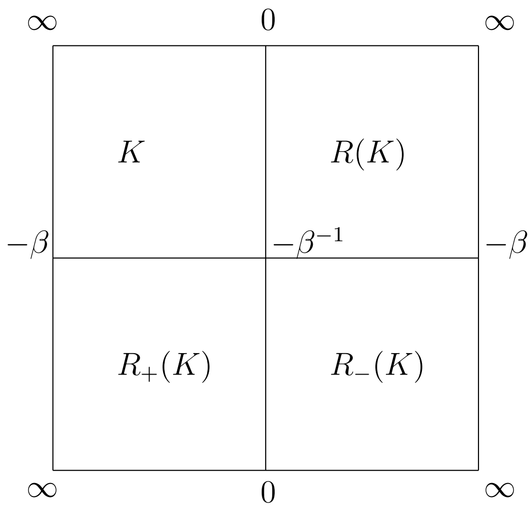
In fig. 2 we indicate the values of the functions $c\wp - \alpha$, $c^3h(\wp)$, z^2 z'^2 in the nodes of Λ_4 in case of $\alpha > \frac{2}{3}$. Up and left to each node, the value of $c\wp - \alpha$ in the node is indicated; up and right, the value of $c^3h(\wp)$ in the node; down and right, the value of z^2; down and left, the value of z'^2:

$c\wp - \alpha$	$c^3h(\wp)$
z'^2	z^2

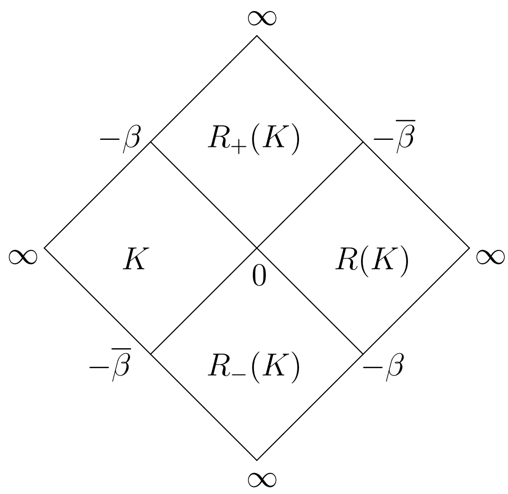
8 Regular solutions with given boundary conditions

Unfix c but assume it does not vanish, and its sign so chosen so as equality (5) is satisfied; assume no longer that α is nonnegative, but demand the exclusion of its singular values

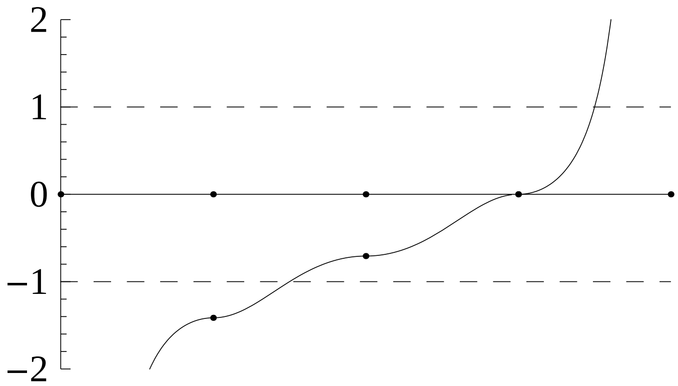
$$c \neq 0, \ \ \alpha \neq \pm\frac{2}{3}.$$



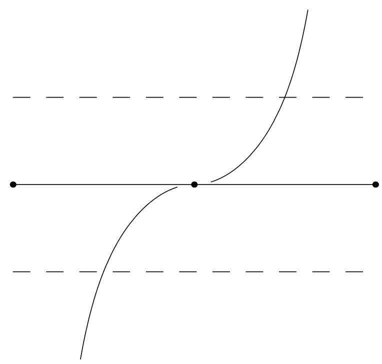
The case of $\alpha = \dfrac{1}{\sqrt{2}}$, $\beta = \sqrt{2}$



The case of $\alpha = 0$, $\beta = i$



Graph of the function $\mathcal{R}$ on four closed internals, corresponding to sides of the half-period square for $\beta = \sqrt{2}$. The derivative function $\mathcal{R}'$ vanishes at the points where the function $\mathcal{R}$ acquires the values $-\beta$, $-\beta^{-1}$ or 0.



Graph of the function $\mathcal{R}$ on two closed internals, corresponding to half-diagonals of the Period square for $\alpha = 0$. The derivative function $\mathcal{R}'$ vanishes whenever the function $\mathcal{R}$ vanishes.

Fig. 1: Period squares and graphs of the function of the function $\mathcal{R}$ in two central cases

∞, 0 $-\delta^2(\Delta+\beta)$, $-\rho_-$	$\lambda+c_2$, ∞ $\delta_-^2(\Delta_- - \beta^{-1})$, $-\rho^{-1}$	∞, 0 $-\delta^2(\Delta-\beta)$, $-\overline{\rho}_-$	$\lambda+c_2$, ∞ $\delta_-^2(\Delta_- + \beta^{-1})$, $-\rho$
$-c_2 d$, $2d(d-\lambda c_2^{-1})$ 0, -1	$c_2 d$, $2d(d+\lambda c_2^{-1})$ $3\alpha-2$, $-\beta^{-1}$	$-c_2 d$, $2d(d-\lambda c_2^{-1})$ 0, -1	$c_2 d$, $2d(d+\lambda c_2^{-1})$ $3\alpha-2$, $-\beta$
0, 0 $-\delta^2(\Delta+\beta)$, $-\overline{\rho}_-$	$\lambda-c_2$, d^2 $\delta_-^2(\Delta_- + \beta^{-1})$, $-\rho^{-1}$	0, 0 $-\delta^2(\Delta-\beta)$, $-\rho_-$	$\lambda-c_2$, d^2 $\delta_-^2(\Delta_- - \beta^{-1})$, $-\rho$
$-c_2 d$, $2d(d-\lambda c_2^{-1})$ ∞, 1	$c_2 d$, $2d(d+\lambda c_2^{-1})$ $3\alpha+2$, 0	$-c_2 d$, $2d(d-\lambda c_2^{-1})$ 0, 1	$c_2 d$, $2d(d+\lambda c_2^{-1})$ $3\alpha+2$, ∞

Fig. 2: The values of the functions $c\wp-\alpha$, $c^3h(\wp)$, z^2 and z'^2 in the nodes of Λ_4 in case of $\alpha > \frac{2}{3}$

We consider in turn the cases of $\alpha > \frac{2}{3}$, $-\frac{2}{3} < \alpha < \frac{2}{3}$ and $\alpha < -\frac{2}{3}$. In all cases we assume that x_0 and x_1 are two units apart from each other

$$x_1 - x_0 = 2.$$

In the first case we regard the middle point x_2 as a fixed point of the shifted lattice $\Lambda_2 + 4^{-1}w_+$, in the second – a point of the shifted lattice $\Lambda + 4^{-1}(w_+ + w_-)$, and in the third – a fixed point of $(\Lambda + 2^{-1}w_+) \cup (\Lambda + 2^{-1}(w_+ + w_-))$:

$$\frac{x_0 + x_1}{2} = x_2 \in \begin{cases} \Lambda_2 + \dfrac{w_+}{4}, \ \forall \alpha \in \left(\dfrac{2}{3}, \infty\right), \\ \Lambda + \dfrac{w_+ + w_-}{4}, \ \forall \alpha \in \left(-\dfrac{2}{3}, \dfrac{2}{3}\right), \\ \left(\Lambda + \dfrac{w_+}{2}\right) \cup \left(\Lambda + \dfrac{w_+ + w_-}{2}\right), \ \forall \alpha \in \left(-\infty, -\dfrac{2}{3}\right). \end{cases}$$

Let $\wp$ be the Weierstrass function of equation (12) with the lattice $\Lambda = \Lambda(\alpha) = \mathbb{Z}w_+ \oplus \mathbb{Z}w_-$,

$$p(x) = \wp(x + x_2),$$

and z – the corresponding to p solution of (8). Then the values of z_0 z_1 of the function z coincide

$$z_0 = z(x_0) = z(x_1) = z_1,$$

in the first two cases, whereas the signs of z_0 z_1 turn out to be opposite to each other in the third

$$z_0 = z(x_0) = -z(x_1) = -z_1.$$

In any case, let

$$r(x) := q\left(\sqrt{c}x\right) = cp\left(\sqrt{c}x\right) - \alpha, \ \ y_i = \frac{x_i}{\sqrt{c}}, \ \ 0 \le i \le 2, \tag{20}$$

and then

$$r'^2 = 4r\left(r + \beta\right)\left(r + \beta^{-1}\right).$$

In calculating the integral in the left hand side of the constraint equation (2), we apply the operator identity

$$K \circ S^{-\alpha} = (I + R) \circ S^{-\alpha} = (S^{-\alpha} \cdot I + S^{-\alpha} \cdot T) \circ S^{-\alpha} = (S^{-\alpha} \circ I + S^{-\alpha} \circ T) = S^{-2\alpha} \circ (I + T)$$

to the function p – a solution of equation (12)

$$p - \frac{\alpha}{c} + c^{-2}\left(p - \frac{\alpha}{c}\right)^{-1} = p + T(p) - \frac{\alpha}{c_2}; \tag{21}$$

and represent the integral in the left hand side of the constraint equation (2) as

$$\int_{x_0}^{x_1} \sqrt{1 + z'^2}\, dx = \int_{x_0}^{x_1} c_2^{-1}(z^2 - \lambda)\, dx = \int_{x_0}^{x_1} \frac{q + q^{-1}}{2}\, dx =$$

$$= \int_{x_0}^{x_1} \frac{cp - \alpha + (cp-\alpha)^{-1}}{2}\, dx = \int_{x_0}^{x_1} c_2\left(p + T(p)\right) - \alpha\, dx =$$

$$= \begin{cases} \int_{x_0}^{x_1} cp\, dx - \alpha = c\left(\zeta(x_0) - \zeta(x_1)\right) - 2\alpha,\ \forall \alpha \in \left(-\frac{2}{3}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \infty\right), \\ c_2\left(\zeta(x_0) + \zeta\left(x_0 + \frac{w_-}{2}\right) - \zeta(x_1) - \zeta\left(x_1 + \frac{w_-}{2}\right)\right) - 2\alpha,\ \forall \alpha \in \left(-\infty, -\frac{2}{3}\right), \end{cases}$$

where ζ is the Weierstrass ζ-function.

8.1 The case of $\alpha > \frac{2}{3}$

Distinguish two subcases: $x_2 \in \Lambda + \frac{w_-}{2} + \frac{w_+}{4}$ $x_2 \in \Lambda + \frac{w_+}{4}$.

Introduce and correspondingly compute the *real* and the *imaginary half-periods* of the lattice $\sqrt{c^{-1}}\Lambda$ of the function r

$$u_+ = \frac{w_+}{2\sqrt{c}},\ u_+ = u_+(\alpha) = \int_0^1 \frac{dx}{\sqrt{x(x+\beta)(x+\beta^{-1})}}.$$

$$u_- = \frac{w_-}{2\sqrt{c}},\ u_- = u_-(\alpha) = -i\int_{-\rho}^{-\beta} \frac{dx}{\sqrt{-x(x+\beta)(x+\beta^{-1})}}.$$

Since

$$2y_2 \in \sqrt{c^{-1}}\Lambda + u_+,$$

then

$$y \in [y_2 - \frac{u_+}{2},\ y_2]$$

implies that

$$r(2y_2 - y) = r(u_+ - y) = r(u_+ + y) = Q \circ r(y) = \frac{1}{r(y)}. \tag{22}$$

We introduce the operator

$$J := M(c)\cdot K = M(c)\cdot(I + R) = I + M(c)\cdot R = I + Q : r \mapsto r + r^{-1}$$

and the map

$$Z_\alpha : y \mapsto \frac{S^{4\lambda} \circ J \circ r(y)}{4(y_2 - y)^2}.$$

The boundary conditions (1) due to (10) and (20) can be expressed as

$$\frac{c(r(y_0) + r(y_1) + 3\alpha)}{4} = z_0^2,$$

and since

$$c = \frac{1}{(y_2 - y_0)^2},\ r(y_0) + r(y_1) = r(y_0) + \frac{1}{r(y_0)} = J \circ r(y_0)$$

they can be written down as

$$Z_\alpha(y_0) = z_0^2.$$

Note that the composition $S^{4\lambda} \circ J \circ r$ is strictly monotonous on the closed interval $[y_2 - 2^{-1}u_+,\ y_2]$, being a composition of three strictly monotonous maps on their corresponding closed intervals.

The subcase $x_2 = \dfrac{2w_- + w_+}{4}$**.** We have

$$y_2 = \frac{2u_- + u_+}{2}, \ \ y_0 = y_2 - \frac{1}{\sqrt{c}}, \ \ y_1 = y_2 + \frac{1}{\sqrt{c}}.$$

The composition $S^{4\lambda} \circ J \circ r$ maps the closed interval $[u_-,\ y_2]$ strictly monotonously onto $[0,\ 3\alpha - 2]$

$$[u_-,\ y_2] \xrightarrow{r} [-\beta,\ -1] \xrightarrow{J} [-3\alpha,\ -2] \xrightarrow{S^{4\lambda}} [0,\ 3\alpha - 2],$$

and, therefore, the map Z_α is strictly monotonous on the semi-closed interval $[u_-,\ y_2)$. Furthermore, the image of the semi-closed interval $[u_-,\ y_2)$ turns out being the whole semi-closed $[0,\ +\infty)$, and we may assert that for each fixed $\alpha > \frac{2}{3}$, the equation

$$Z_\alpha(y) - z_0^2 = 0 \tag{23}$$

has a unique root $y = y_0$ on the semi-closed interval $[u_-,\ y_2)$.

In case of $z_0 = 0$, the root of (23) on the indicated semi-closed interval is necessarily $y = y_0 = u_-$ implying that $y_1 = u_- + u_+$ and the difference $y_1 - y_0 = 2\sqrt{c^{-1}}$ must coincide with the real half-period u_+, and, therefore,

$$c = 4\left(\int_0^1 \frac{dx}{\sqrt{x(x+\beta)(x+\beta^{-1})}}\right)^{-2}.$$

We present graphs of solutions of equation (8) with boundary conditions $x_0 = -1,\ x_1 = 1\ \ z_0 = z_1$, corresponding to distinct number of waves and distinct values of α. Fig. 4 represents three solutions of equation (8) with boundary conditions $x_0 = -1,\ x_1 = 1,\ z_0 = 0 = z_1$, corresponding to the value of $\alpha = \frac{3}{4}$. Fig. 5 represents three solutions of equation (8) and their continuations for boundary conditions $x_0 = -1,\ x_1 = 1,\ z_0 = 1 = z_1$, corresponding to the value of $\alpha = 3$. Fig. 6 represents three solutions of equation (8) and their continuations for boundary conditions $x_0 = -1,\ x_1 = 1,\ z_0 = 1 = z_1$, corresponding to values of $\alpha = 1$, 2 and 3.

The subcase $x_2 = \dfrac{w_+}{4}$**.** We have

$$y_2 = \frac{u_+}{2}, \ \ y_0 = y_2 - \frac{1}{\sqrt{c}}, \ \ y_1 = y_2 + \frac{1}{\sqrt{c}}.$$

But in this case, the map Z_α is not monotonous on the interval $(0,\ y_2)$, which need not contain the root of equation (23). However, for each fixed $\alpha > \frac{2}{3}$ and fixed $z_0 > 0$ the equation (23) possesses no more than two roots on $(0,\ y_2)$.

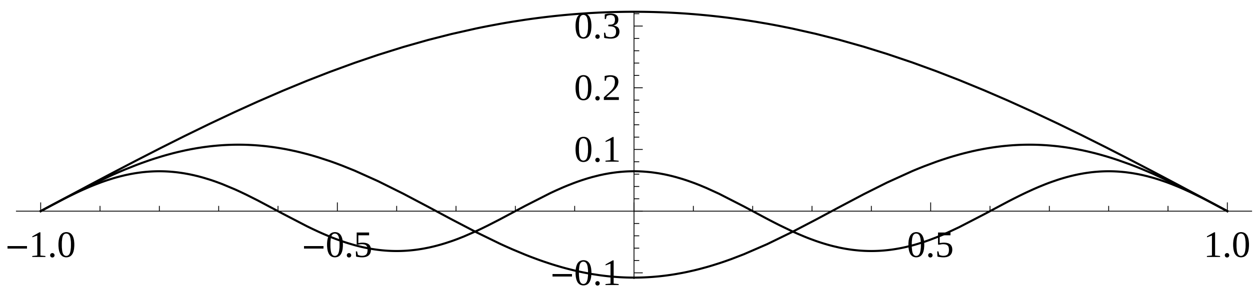


Fig. 4

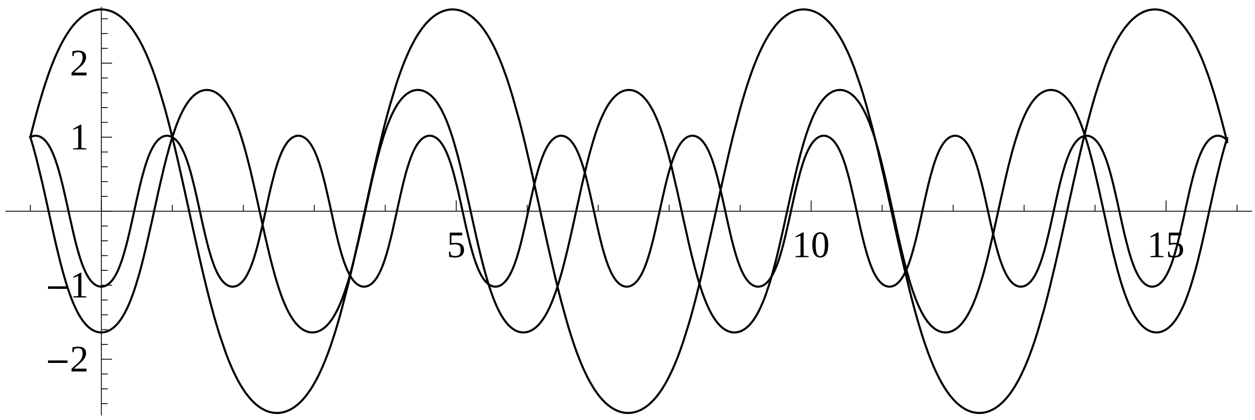


Fig. 5

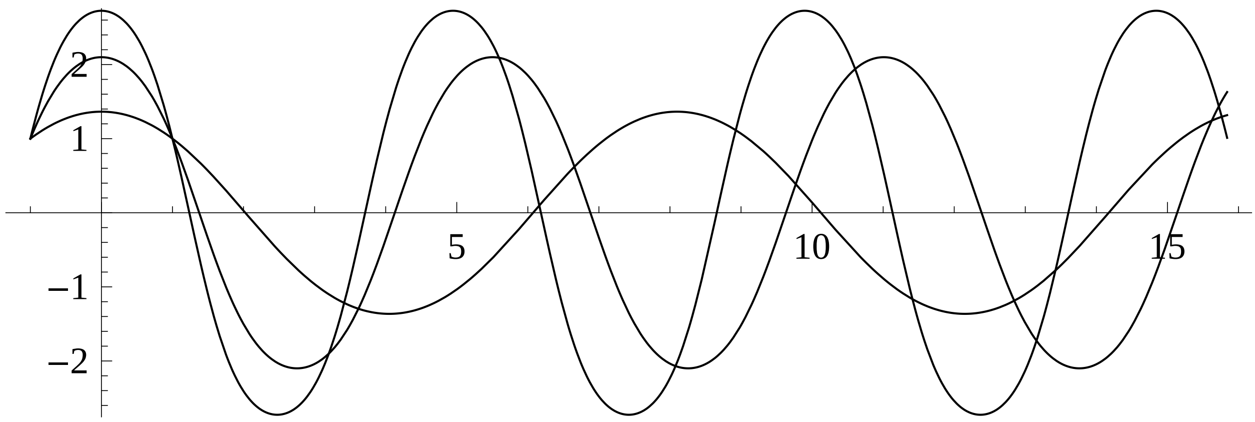


Fig. 6

8.2 The case of $-\frac{2}{3} < \alpha < \frac{2}{3}$

We now introduce and calculate *the real half-diagonal*

$$u_+ = \frac{w_+ + w_-}{2\sqrt{c}}, \;\; u_+ = u_+(\alpha) = \int_0^1 \frac{dx}{\sqrt{x(x+\beta)(x+\overline{\beta})}}$$

and *the imaginary half-diagonal*

$$u_- = \frac{w_+ - w_-}{2\sqrt{c}}, \;\; u_- = u_-(\alpha) = -i\int_{-1}^0 \frac{dx}{\sqrt{-x(x+\beta)(x+\overline{\beta})}}.$$

We have

$$y_2 = \frac{u_+}{2}, \;\; y_0 = y_2 - \frac{1}{\sqrt{c}}, \;\; y_1 = y_2 + \frac{1}{\sqrt{c}}.$$

The subcase of $0 < \alpha < \frac{2}{3}$. As in the previous subcase, for each fixed α, $0 < \alpha < \frac{2}{3}$, and fixed $z_0 > 0$ there exist no more than two values for $c > 0$, for whom the boundary conditions

$$x_0 = -1, \;\; x_1 = 1, \;\; z_0 = z_1$$

hold.

The subcase of $\alpha = 0$. The moduli of the real and the imaginary half-diagonals coincide with each other in this central case

$$u_+ = -iu_- = \int_0^1 \frac{dx}{\sqrt{x(x^2+1)}} \approx 1.85407467730,$$

and the length of the side of the period square coincides with the semi-length of the lemniscate of Bernoulli, the focal distance of which is two

$$\sqrt{2}\int_0^1 \frac{dx}{\sqrt{x(x^2+1)}} = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = k\pi \approx 2.62205755429,$$

where k is Gauss' constant – the inverse of the arithmetic-geometric mean of 1 and $\sqrt{2}$. However, we would not have pointed out this case merely for historical reasons. The solutions for the case of $\alpha = 0$ are the only unconditional extremals of the functional U, which are not simultaneously extremals of the length functional L.

For $z_0 < d_0 \approx 2.36746199454$ the equation (23) has no roots on the interval $(0,\, y_2)$. It possesses a unique root in that interval for $z_0 = d_0$, and possesses exactly two roots for $z_0 > d_0$.

The subcase of $-\frac{2}{3} < \alpha < 0$. Can be brought back to the subcase of $0 < \alpha < \frac{2}{3}$, if one makes use, for a fixed $c > 0$, of the identities

$$\Lambda(\alpha) = i\Lambda(-\alpha)$$

and

$$\wp(x; \Lambda(\alpha)) = -\wp(ix;\ \Lambda(-\alpha)); \tag{24}$$

and views the solutions on the imaginary axis. As with the subcase of $0 < \alpha < \frac{2}{3}$, for each fixed α, $-\frac{2}{3} < \alpha < 0$, and fixed imaginary $z_0 \neq 0$ there are no more than two solutions, for which the conditions

$$x_0 = -i,\ \ x_1 = i,\ \ z_0 = z_1$$

are satisfied.

8.3 The case of $\alpha < -\frac{2}{3}$

This case, utilizing (24), can be viewed as another subcase of $\alpha > \frac{2}{3}$.

The subcase $x_2 \in \left(\Lambda + \dfrac{w_-(|\alpha|)}{2}\right) \cup \left(\Lambda + \dfrac{w_-(|\alpha|) + w_+(|\alpha|)}{2}\right)$**.** In viewing the solutions on the imaginary axis, the difference between the points x_0 and x_1 is pure imaginary, and the signs of z_0 and z_1 are opposite to each other,

$$x_0 = -i,\ \ x_1 = i,\ \ z_0 = -z_1.$$

We also point out that

$$u_+(\alpha) = -iu_-(-\alpha),\ \ u_-(\alpha) = iu_+(-\alpha).$$

And if we regard the sign of z_0 fixed, say $iz_0 > 0$, then equation (23) has no more than one root on the interval $(0,\ y_2)$. More precisely, equation (23), on this interval, has no roots for $0 < iz_0 \leq d(\alpha)$, and possesses exactly one root when $iz_0 > d(\alpha)$.

Here we may assume that

$$Z_\alpha(y_2) = d^2(\alpha),$$

since $Z_\alpha(y) \to d^2(\alpha)$ for $y \to y_2$. Moving on to discussing non regular solutions, we begin with the case of $\alpha < -\frac{2}{3}$, corresponding to limit solutions.

9 Non regular solutions with given boundary conditions

9.1 The case of $\alpha < -\frac{2}{3}$

If in case of $\alpha < -\frac{2}{3}$ the value of z_0 is critical, that is

$$iz_0 = d = d(\alpha),$$

then a solution we ought to be regard is the linear solution

$$z = dx,$$

which is a limit solution of the form (18).

9.2 The case of $\alpha = -\frac{2}{3}$

Here

$$d\left(-\frac{2}{3}\right) = 0.$$

If $z_0 = z_1 > 0$ then, along with the one-sided marginal solution corresponding to the case of $\alpha = -\frac{2}{3}$, a solution we ought to regard is the solution

$$z = z_0,$$

which is a limit solution of the form (18), namely, a horizontal non-axial solution.

If the signs of z_0 and z_1 are opposite to each other, $z_0 = -z_1 > 0$, then the corresponding two-sided marginal solution turns out being the unique solution corresponding to the case of $\alpha = -\frac{2}{3}$.

9.3 The case of $\alpha = \frac{2}{3}$

This special unbounded solution satisfies the boundary conditions

$$x_0 = -1, \;\; x_1 = 1, \;\; z_0 = z_1,$$

with the additional condition

$$z_0 \geq d_+ := 2x^{-1}\sec(x)\big|_{x\tan(x)=1,\; 0<x<2^{-1}\pi} \approx \; 3.5644502804.$$

Here we define the real half-period u_+ for $\alpha = \frac{2}{3}$

$$u_+\left(\frac{2}{3}\right) := \lim_{\alpha\to\frac{2}{3}} u_+(\alpha) = \lim_{\beta\to 1}\int_0^1 \frac{dx}{\sqrt{x(x+\beta)(x+\beta^{-1})}} =$$

$$= \int_0^1 \frac{dx}{\sqrt{x}(x+1)} = 2\arctan\left(\sqrt{x}\right)\Big|_0^1 = \frac{\pi}{2}.$$

10 Four solutions families

10.1 A family of vertical and horizontal axial solutions

The vertical solutions correspond to the limiting value $\alpha = \pm\infty$, whereas the horizontal axial solutions correspond to the critical values $\alpha = \pm\frac{2}{3}$. The magnitude of the tension is undetermined for this solutions family.

10.2 A solutions family with a fixed magnitude of horizontal component of tension

Fig. 7 exhibits equilibrium forms for the same nonzero magnitude of the horizontal component of the tension, corresponding to subsequent values of the modulus of α

$$0 < \frac{2}{3} < \frac{1}{\sqrt{2}} < 1$$

.

All solutions of this family have $|c| = 1$, which corresponds to $|\tau| = 2^{-1}|\sigma|$. In particular, the horizontal axial solution has a nonzero tension, the magnitude of which is $2^{-1}|\sigma|$.

10.3 A family of unbounded solutions sharing two vertical asymptotes

Fig. 8 exhibits equilibrium forms whose vertical asymptotes are $x = 0$ and $x = 1$, corresponding to subsequent values of the modulus of α

$$0 < \frac{19}{29} < \frac{59}{89} < \frac{1999}{2999} < \frac{2}{3} < \frac{401}{601} < \frac{41}{61} < \frac{13}{19}.$$

10.4 A family of even and odd unbounded solutions passing through $(-1, 1)$

Fig. 9 exhibits nine equilibrium curves, passing through $(-1, 1)$, corresponding to subsequent values of the modulus of α

$$\frac{19}{29} < \frac{59}{89} < \frac{1999}{2999} < \frac{401}{601} < \frac{41}{61} < \frac{13}{19}.$$

The six even equilibrium curves correspond to the two values of the modulus of α, less than $\frac{2}{3}$, whereas the three odd curves correspond to the two values of the modulus of α greater than $\frac{2}{3}$.

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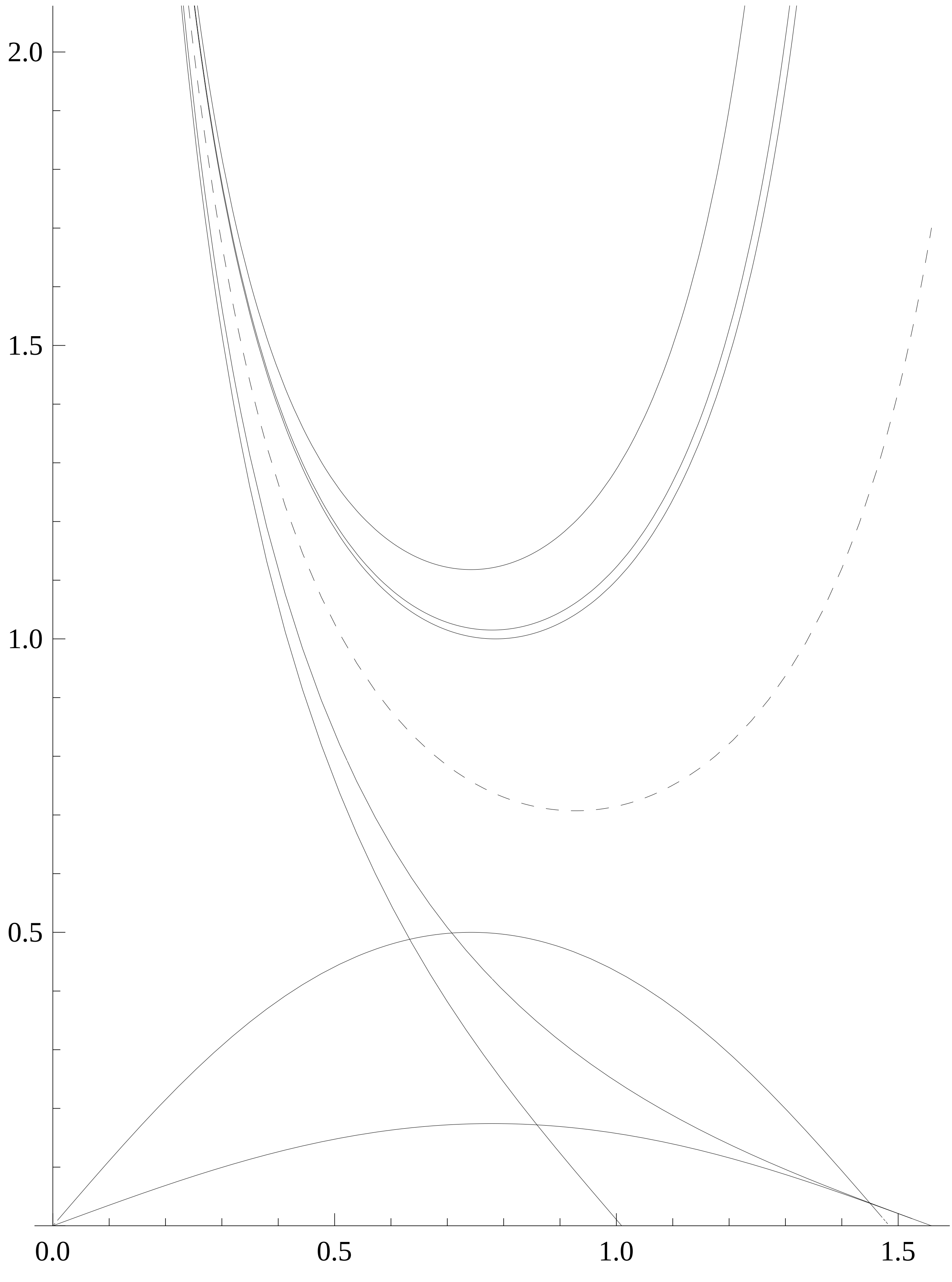

Fig. 7

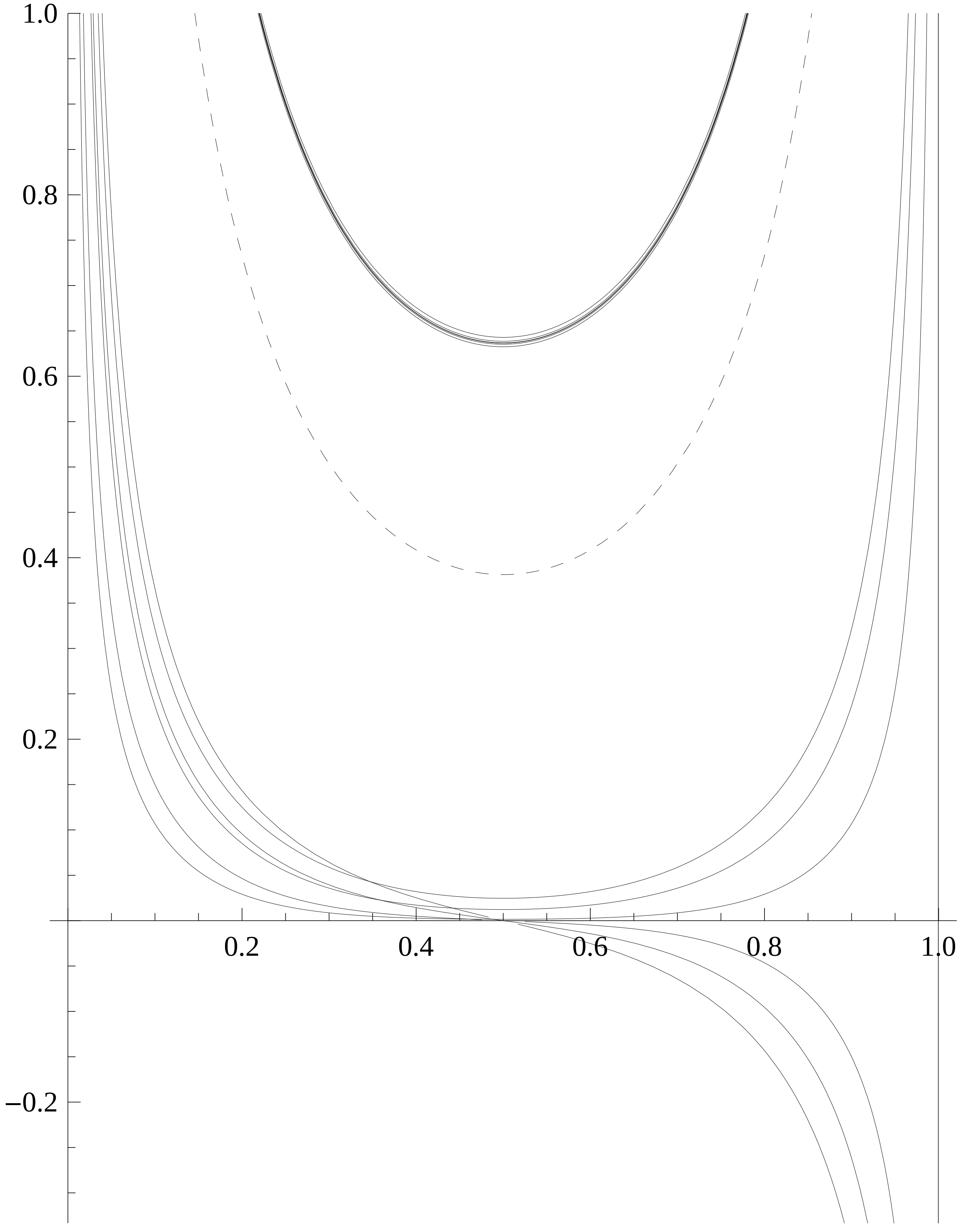

Fig. 8

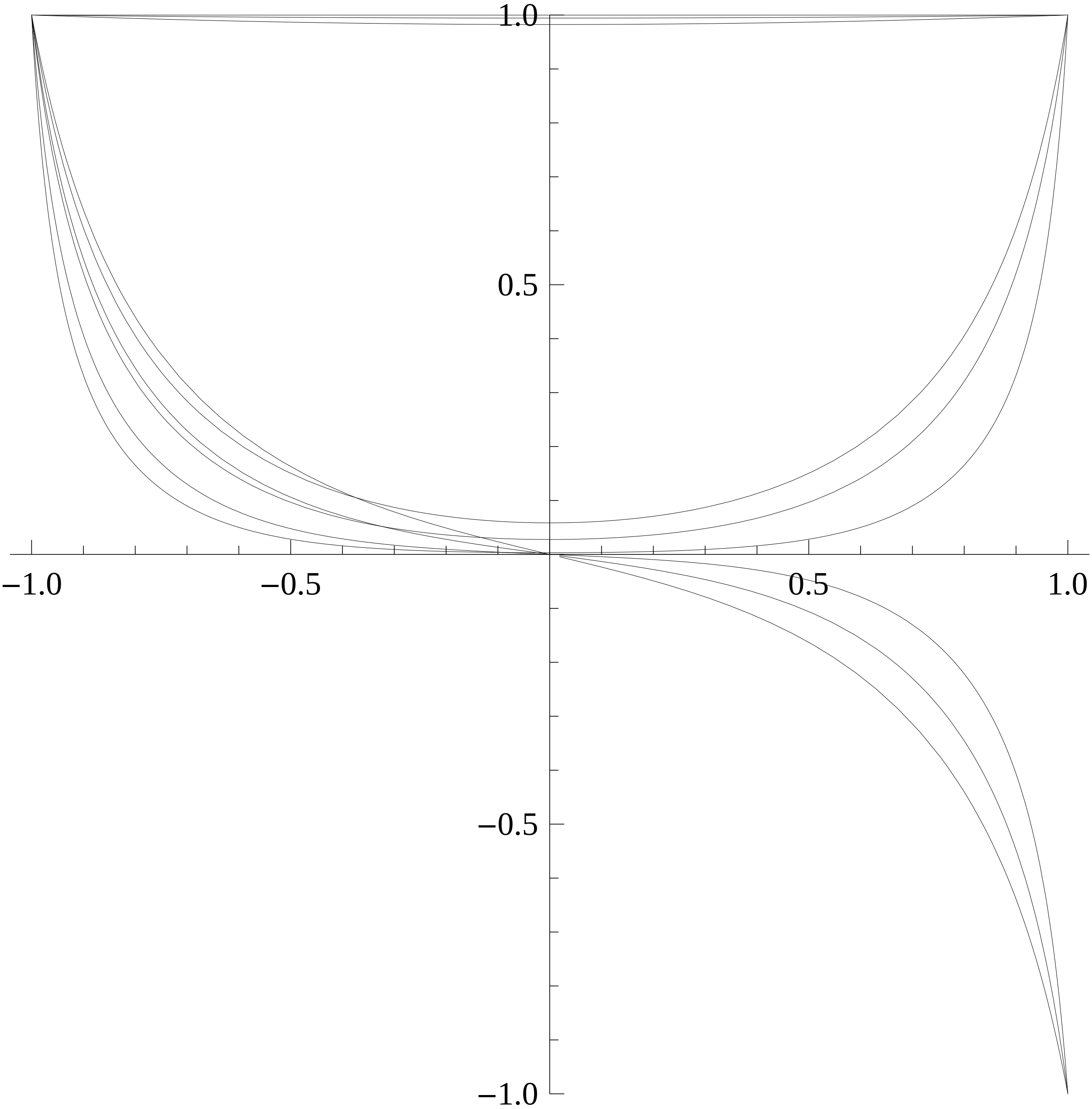

Fig. 9